SYNTHESIS OF MINIMAL SENSITIVITY SAMPLED-DATA CONTROL SYSTEMS

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ABSTRACT

A frequency domain method of synthesizing the minimal sensitivity sampled-data control systems is developed. A minimal sensitivity system is the one having a shortest sensitivity polynomial. The method is simple, easy to apply, and is not restricted to the order of the plant. Several discussions are made. Three examples are given to demonstrate the effectiveness of the method.

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INTRODUCTION

One important feature of a linear sampled-data control system, which is not possessed by its continuous-data counterpart, is that it can be designed to achieve an error-free finite settling time in response to a certain class of inputs (1-5)*. However, engineers are reluctant to enjoy this feature because of the general feeling that sampled-data systems of this type suffer from infinite pole-sensitivity with respect to system parameter variations.

The error-free finite settling time property will be called "deadbeat" property in this paper, and a system possessing this property will be called a "deadbeat system." (Figure 1).

The word infinite pole-sensitivity is rather misleading, since it is associated with a particular definition of pole-sensitivity which may not be suitable for certain types of systems. It is well known that a deadbeat sampled-data system has a multiple pole at the origin of the z-plane. The use of the pole-sensitivity defined by

$$S_x^{s_j} = \frac{ds_j}{dx} , \qquad (1)$$

where s_j is the closed-loop pole and x is a changing parameter, leads to infinite pole-sensitivity. The real effect on the system pole s_j due to

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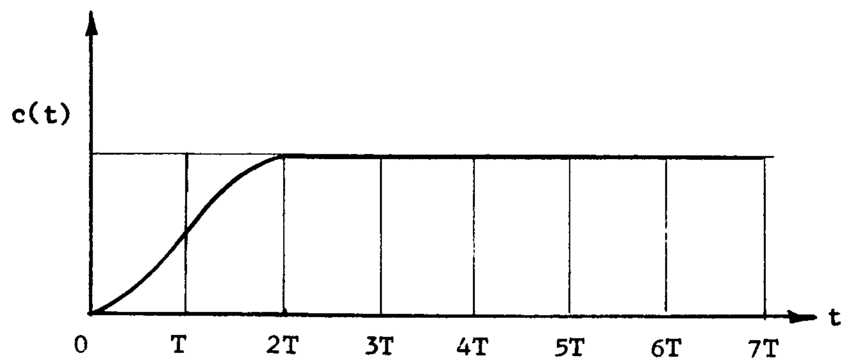
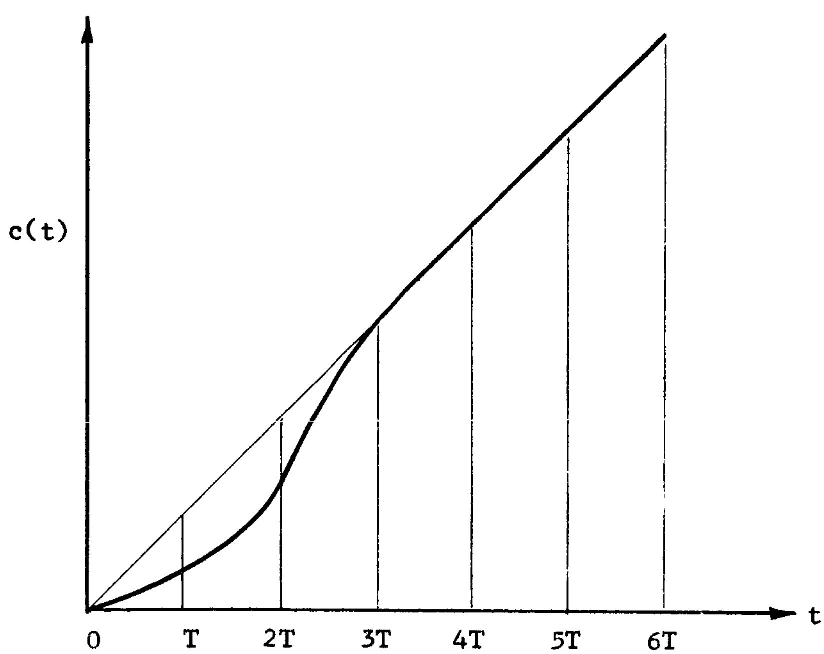
*Numbers in parentheses represent similarly numbered entries in the "List of References."



(a) Step response



(b) Ramp response

Figure 1. Deadbeat response.

the parameter change is not only seldom infinite but may not even be large. It has been shown (6) that under this condition a suitable pole-sensitivity would be

$$S_x^{s_j} = \frac{(ds_j)^m}{dx} \tag{2}$$

where m is the multiplicity of the multiple pole s_j. This definition of pole-sensitivity results in a finite sensitivity value and has been demonstrated reliable.

Considerable amount of attention has been given to the sensitivity problems in control systems in recent years. Both frequency domain and time domain approaches have been attempted.

However, very little attention has been given to the design of low sensitivity deadbeat sampled-data control systems. Although a deadbeat system loses its deadbeat property for any change of system parameter, systems with lower sensitivity will have a response closer to deadbeat. This paper presents a frequency domain (z-transform) method for designing minimal sensitivity deadbeat sampled-data control systems.

Instead of using the pole-sensitivity defined by Equation (2), a different concept of sensitivity is used for our problem (9). Let C(z) be the output of a deadbeat sampled-data system responding to a certain class of input, R(z). A differential change of a system parameter causes a differential change in output response, which is denoted by dC(z). Note that dC(z) is a polynomial in z^{-1}, which will be called the "sensitivity polynomial." A "minimal sensitivity" system is defined as a system having the shortest sensitivity polynomial.

From the topological point of view, the input-output relation and the sensitivity of a feedback system can be controlled simultaneously by using a two-degree-of-freedom structure having a cascade controller and a feedback controller (7,8). In the following, a method of designing the minimal sensitivity sampled-data control system will be developed. The method of design is simple, and several examples will be given for illustration.

CHAPTER II

DEVELOPMENT OF THE METHOD

Consider a sampled-data control system having two digital compensators, $D_1(z)$ and $D_2(z)$, and a plant, $G(z)$, as shown in Figure 2. The closed-loop transfer function of the system is

$$H(z) = \frac{C(z)}{R(z)} = \frac{D_1(z)G(z)}{1 + D_1(z)D_2(z)G(z)} \quad . \tag{3}$$

It is easily shown that the differential change of the output response is related to the differential change of the plant by

$$\frac{dC(z)}{C(z)} = \left[1 - D_2(z)H(z) \right] \frac{dG(z)}{G(z)} \quad , \tag{4}$$

or

$$dC(z) = R(z)H(z) \left[1 - D_2(z)H(z) \right] \frac{dG(z)}{G(z)} \quad . \tag{5}$$

This is the "sensitivity polynomial."

The plant can be expressed in the general form

$$G(z) = \frac{K\, z^{-\delta} \prod_j (1 - z_j z^{-1})^{m_j}}{\prod_i (1 - p_i z^{-1})^{n_i}} \tag{6}$$

where m_j and n_i are the orders of the zeros z_j and of the poles p_i respectively, K is the gain, and $z^{-\delta}$ is the plant transport lag. It can be shown that the variations in the plant with respect to its gain, pole and zero is

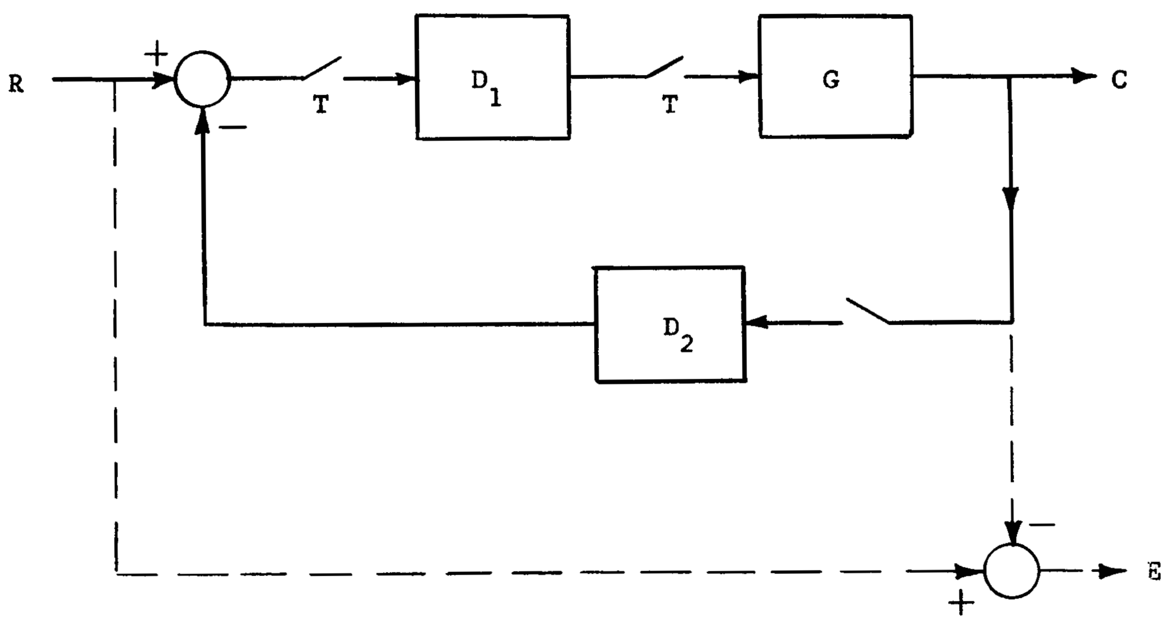


Figure 2. Two-controller sampled-data control system.

$$\frac{dG(z)}{G(z)} = \frac{Q(z)}{P(z)} = \frac{dK}{K} + \sum_i \frac{n_i z^{-1} dp_i}{1 - p_i z^{-1}} - \sum_j \frac{m_j z^{-1} dz_j}{1 - z_j z^{-1}} \tag{7}$$

where Q(z) and P(z) are polynomials in z^{-1}. The reference input takes the general form

$$R(z) = \frac{A(z)}{(1 - z^{-1})^m} \tag{8}$$

with A(z) being a finite polynomial and having no zero at z = 1.

Referring to Equation (5), R(z) and G(z) are in general given, and H(z) is completely determined by the required deadbeat property of the system and its physical realizability conditions (1,2). This leaves only $D_2(z)$ which can be adjusted for the shortest dC(z).

Let U(z) and V(z) be the numerator and the denominator polynomial of $D_2(z)$, respectively, i.e.

$$D_2(z) = \frac{U(z)}{V(z)} \quad . \tag{9}$$

The physical realizability requires that V(z) must contain a constant term. The closed-loop transfer function of a deadbeat system has the form

$$H(z) = z^{-\delta} F(z) \tag{10}$$

where $z^{-\delta}$ is the plant transport lag and F(z) is a finite polynomial with a constant term. Substituting Equations (8), (9) and (10) into Equation (5), gives

$$dC(z) = \frac{A(z)}{(1 - z^{-1})^m} H(z) \left[1 - \frac{U(z) z^{-\delta} F(z)}{V(z)} \right] \frac{Q(z)}{P(z)} \quad . \tag{11}$$

Since there may be cancellations among A(z), H(z), P(z) and $(1 - z^{-1})^m$, let

$$\frac{N(z)}{M(z)} = \frac{A(z)H(z)}{(1 - z^{-1})^m P(z)} \tag{12}$$

where N(z) and M(z) are finite polynomials in z^{-1} and are relatively prime. Then Equation (11) becomes

$$dC(z) = \frac{N(z)}{M(z)}\left[1 - \frac{U(z)z^{-\delta}F(z)}{V(z)}\right]Q(z) \quad . \tag{13}$$

Examining Equation (13) reveals that dC(z) is made the shortest when

$$V(z) = F(z) \tag{14}$$

and when U(z) is so chosen such that

$$S(z) = \frac{1 - U(z)z^{-\delta}}{M(z)} \tag{15}$$

is the shorted polynomial in z^{-1}. Equations (14) and (15) may be called the "minimal sensitivity condition."

Once V(z) and U(z) are found $D_2(z)$ is determined. Then $D_1(z)$ is given by

$$D_1(z) = \frac{H(z)}{G(z)\left[1 - D_2(z)H(z)\right]} \quad . \tag{16}$$

CHAPTER III

DESIGN PROCEDURE

The theory of designing the closed-loop transfer function, H(z), for a deadbeat system is available in literature (1,2). Here, a complete design procedure, for the minimal sensitivity deadbeat system, is outlined.

Given the plant and the input, Equations (6) and (8), the design procedure is as follows:

Step I. Let

$$H(z) = z^{-\delta} \prod_j (1 - z_j z^{-1})^{m_j} (a_o + a_1 z^{-1} + \dots + a_{m-1} z^{-m+1}) \tag{17}$$

and solve for $a_o, a_1, \dots, a_{m-1}$ from the condition

$$\left.\begin{aligned} &H(1) = 1 \\ &H'(1) = 0 \\ &\quad\cdot \\ &\quad\cdot \\ &\quad\cdot \\ &H^{(m-1)}(1) = 0 \end{aligned}\right\} \tag{18}$$

where the derivatives of H(z) are taken with respect to z^{-1}.

Step II. Determine $\frac{dG(z)}{G(z)}$ as given by Equation (7) and determine M(z) using Equation (12).

Step III. Determine V(z) and U(z) from the minimal sensitivity condition, Equations (14) and (15).

Step IV. The two controllers are given by

$$\left.\begin{aligned} D_2(z) &= \frac{U(z)}{V(z)} \\ &\text{and} \\ D_1(z) &= \frac{H(z)}{G(z)\left[1 - D_2(z)H(z)\right]} \end{aligned}\right\} \tag{19}$$

CHAPTER IV

REMARKS

Although the differential sensitivity polynomial, $dC(z)$, can be made a finite polynomial in z^{-1} by the above design technique, the output deviation in a real system is in general an infinite polynomial. This is due to the fact that for an incremental change of the plant parameters, the change of the output as given by Equation (5), is only an approximation. However, if the plant variation is not severe, the actual polynomial of the output deviation will approximate the sensitivity polynomial.

The system input is not restricted to the form shown in Equation (8). For example, if the system input is in the form

$$R(z) = \frac{A(z)}{(1 - az^{-1})^{\alpha}(1 - bz^{-1})^{\beta}} \quad , \tag{20}$$

then the system transfer function becomes

$$\underline{H(z)} = z^{-\delta} \prod_i (1 - z_i z^{-1})^{m_i} (a_o + a_1 z^{-1} + \dots a_{\alpha+\beta-1} z^{-\alpha-\beta+1}) \quad . \tag{21}$$

The coefficients, a_o, a_1, ... $a_{\alpha+\beta-1}$, can be obtained by a procedure, similar to Equation (18), as follows:

$$\left.\begin{aligned} H(a) &= 1 \quad , \\ H'(a) &= 0 \quad , \\ &\vdots \\ H^{(\alpha-1)}(a) &= 0 \quad . \end{aligned}\right\} \tag{22a}$$

$$\left.\begin{aligned} &H(b) = 1 \quad , \\ &H'(b) = 0 \quad , \\ &\quad \vdots \\ &H^{(\beta-1)}(b) = 0 \quad . \end{aligned}\right\} \qquad (22b)$$

However, under this condition, the output is in general not ripple-free.

One word should be said about the controllability of the system. The plant is controllable if it is free of poles with equal real parts whose imaginary parts are separated by an integral multiple of the sampling frequency.

CHAPTER V

EXAMPLES

Three examples are given to demonstrate the effectiveness of the proposed method.

I. EXAMPLE 1

Given the plant

$$G(z) = \frac{0.3679z^{-1}(1 + 0.7183z^{-1})}{(1 - z^{-1})(1 - 0.3689z^{-1})} \quad ,$$

design the minimal sensitivity deadbeat system for a step input, where the plant gain is the varying parameters.

Step I. Take

$$H(z) = z^{-1}(1 + 0.7183z^{-1})a_o$$

Letting $H(1) = 1$, gives $a_o = 0.5820$. Hence,

$$H(z) = 0.582z^{-1}(1 + 0.7183z^{-1}) \quad ,$$

$$E(z) = 1 + 0.418z^{-1} \quad , \quad \text{and}$$

$$\frac{dG(z)}{G(z)} = \frac{dK}{K} \quad , \text{ a constant.}$$

Step II. Using Equation (12),

$$\frac{N(z)}{M(z)} = \frac{0.5825z^{-1}(1 + 0.7183z^{-1})}{(1 - z^{-1})}$$

where

$$M(z) = 1 - z^{-1}.$$

Step III. Let

$$V(z) = 0.582(1 + 0.7183z^{-1}).$$

By taking $U(z) = 1$,

$$S(z) = \frac{1 - U(z)z^{-1}}{1 - z^{-1}} = 1 \quad ,$$

the shortest polynomial.

Step IV. Using Equation (19),

$$D_2(z) = \frac{1.7183}{1 + 0.7183z^{-1}}$$

and

$$D_1(z) = 1.582(1 - 0.3679z^{-1}).$$

It would be interesting to compare the performance of the two-controller system to that of one-controller system, Figure 3. The design of the one-controller system is known (1). Under the nominal condition, both systems have identical input-output response.

Consider a gain change of 50 percent, the varied plant becomes

$$G_v(z) = \frac{0.5518z^{-1}(1 + 0.7183z^{-1})}{(1 - z^{-1})(1 - 0.3679z^{-1})} \quad .$$

The subscript "v" denotes the varied function. Table I contains the comparison of the two systems under varied conditions. For the ease of comparison, Figure 4 shows the system error for both systems under nominal conditions and varied conditions. The merit of two-controller system is evident.

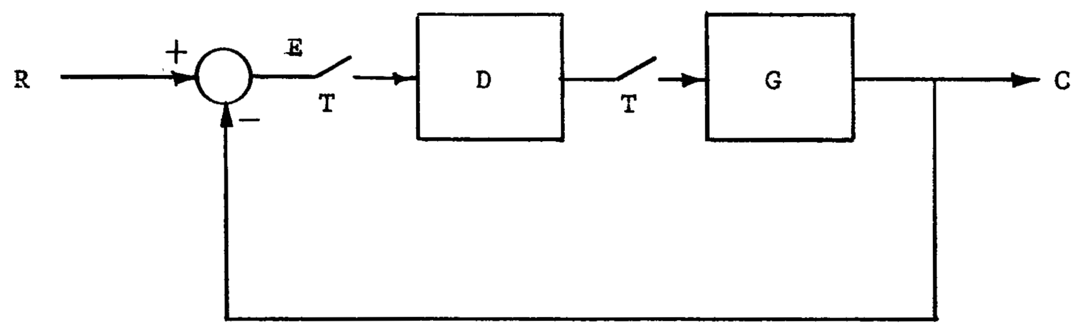


Figure 3. One-controller sampled-data control system.

TABLE I

COMPARISON FOR EXAMPLE 1

Two-Controller System

$$D_1(z) = 1.582(1 - 0.3679z^{-1})$$

$$D_2(z) = \frac{1.7183}{1 + 0.7183z^{-1}}$$

One-Controller System

$$D(z) = \frac{1.582(1 - 0.3679z^{-1}}{1 + 0.418z^{-1}}$$

Gain Variation by 50%:

$$G_v(z) = \frac{0.5518z^{-1}(1 + 0.7183z^{-1})}{(1 - z^{-1})(1 - 0.3679z^{-1})}$$

Two-Controller System

$$H_v(z) = \frac{0.873z^{-1}(1 + 0.7183z^{-1})}{1 + 0.5\ z^{-1}}$$

$$E_v(z) = 1 + 0.127z^{-1} - 0.064z^{-2} + 0.0307z^{-3}$$
$$-0.154z^{-4} + 0.0077z^{-5} - 0.00382z^{-6} \text{---}$$

One-Controller System

$$H_v(z) = \frac{0.873z^{-1}(1 + 0.7183z^{-1})}{1 + 0.291z^{-1} + 0.209z^{-2}}$$

$$E_v(z) = 1 + 0.1273z^{-1} + 0.246z^{-2} + 0.046z^{-3}$$
$$+0.038z^{-4} - 0.0207z^{-5} - 0.058z^{-6} \text{---}$$

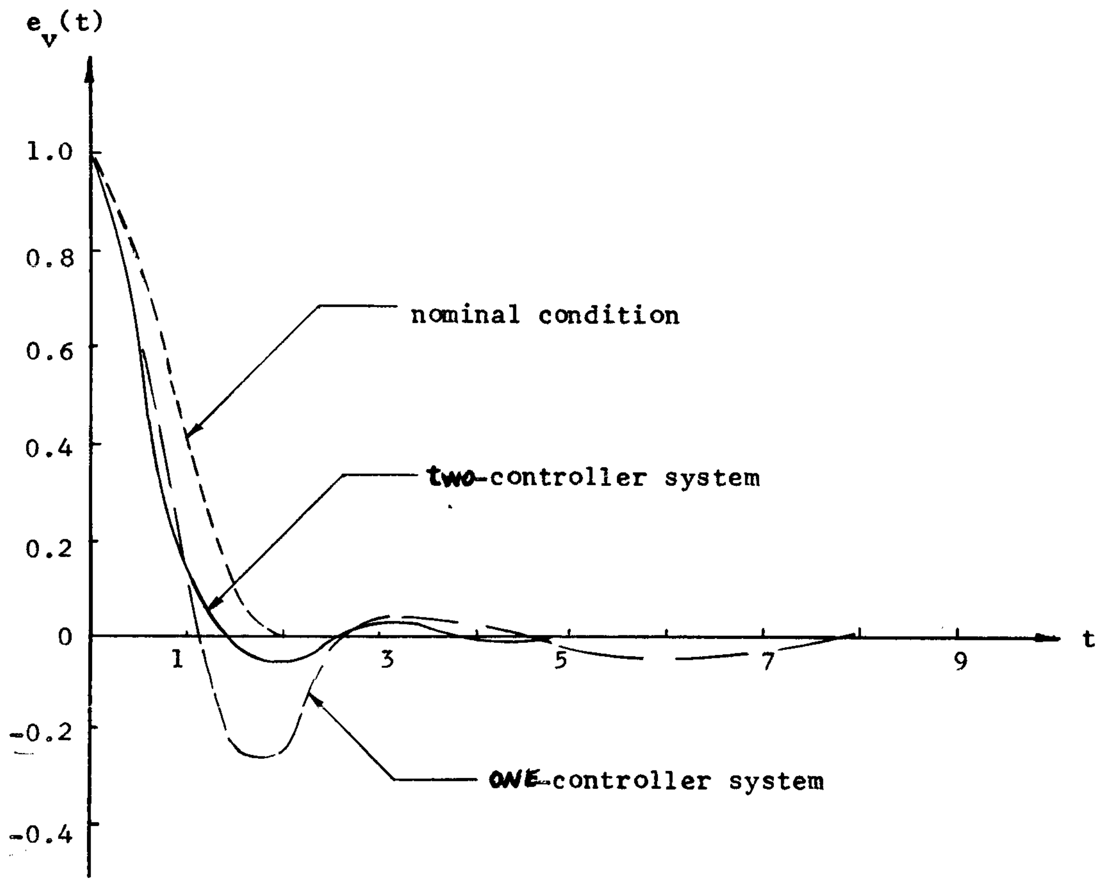


Figure 4. System errors for Example 1.

II. EXAMPLE 2

The problem is the same as Example 1, except now the plant pole at z = 1 is the varying parameter. Here

$$\frac{dG(z)}{G(z)} = \frac{0.5z^{-1}}{1 - z^{-1}} .$$

The design result of the two-controller system and the comparison to the one-controller system is tabulated in Table II. Figure 5 shows the comparison of the system errors. Note that G(z), H(z) and E(z), the nominal condition functions, are the same as in Example 1.

III. EXAMPLE 3

In a practical system, the change of a single plant parameter may cause a simultaneous change of gain, poles and zeros of the plant. This example will illustrate this case.

Consider a separately excited dc motor driving a load. The transfer function between the applied armature voltage V and the motor shaft position θ is

$$\frac{\theta}{V} = \frac{\dfrac{K_T}{J R_a}}{s\left[s + \dfrac{1}{J}\left(B + \dfrac{K_T K_e}{R_a}\right)\right]}$$

where

J = 443.0 slug-ft^2, armature-load inertia ,

B = 160 lb-ft/rad-sec, friction coefficient ,

TABLE II

COMPARISON FOR EXAMPLE 2

Two-Controller System

$$D_1(z) = \frac{1.582(1 - 0.3679z^{-1})}{(1 - z^{-1})}$$

$$D_2(z) = \frac{3.4364(1 - 0.5z^{-1})}{1 + 0.7183z^{-1}}$$

One-Controller System

$$D(z) = \frac{1.582(1 - 0.3679z^{-1})}{1 + 0.418z^{-1}}$$

Pole Variation by 50%:

$$G_v(z) = \frac{0.368z^{-1}(1 + 0.7183z^{-1})}{(1 - 1.5z^{-1})(1 - 0.3679z^{-1})}$$

Two-Controller System

$$H_v(z) = \frac{0.582z^{-1}(1 + 0.7183z^{-1})}{1 - 0.5z^{-1} + 0.5z^{-2}}$$

$$\begin{aligned} E_v(z) = 1 &+ 0.418z^{-1} - 0.291z^{-2} - 0.3545z^{-3} - 0.0318z^{-4} \\ &+ 0.1614z^{-5} + 0.0966z^{-6} - 0.0328z^{-7} - 0.0647z^{-8} \\ &- 0.0162z^{-9} + 0.0240z^{-10} - 0.02z^{-11} --- \end{aligned}$$

One-Controller System

$$H_v(z) = \frac{0.582z^{-1}(1 + 0.7183z^{-1})}{1 - 0.5z^{-1} - 0.209z^{-2}}$$

$$\begin{aligned} E_v(z) = 1 &+ 0.418z^{-1} - 0.2910z^{-2} - 0.7671z^{-3} - 1.1534z^{-4} \\ &- 1.4460z^{-5} - 1.6731z^{-6} - 1.8502z^{-7} - 1.9854z^{-8} \\ &- 2.0896z^{-9} - 2.170z^{-10} - 2.232z^{-11} - 2.2798z^{-12} --- \end{aligned}$$

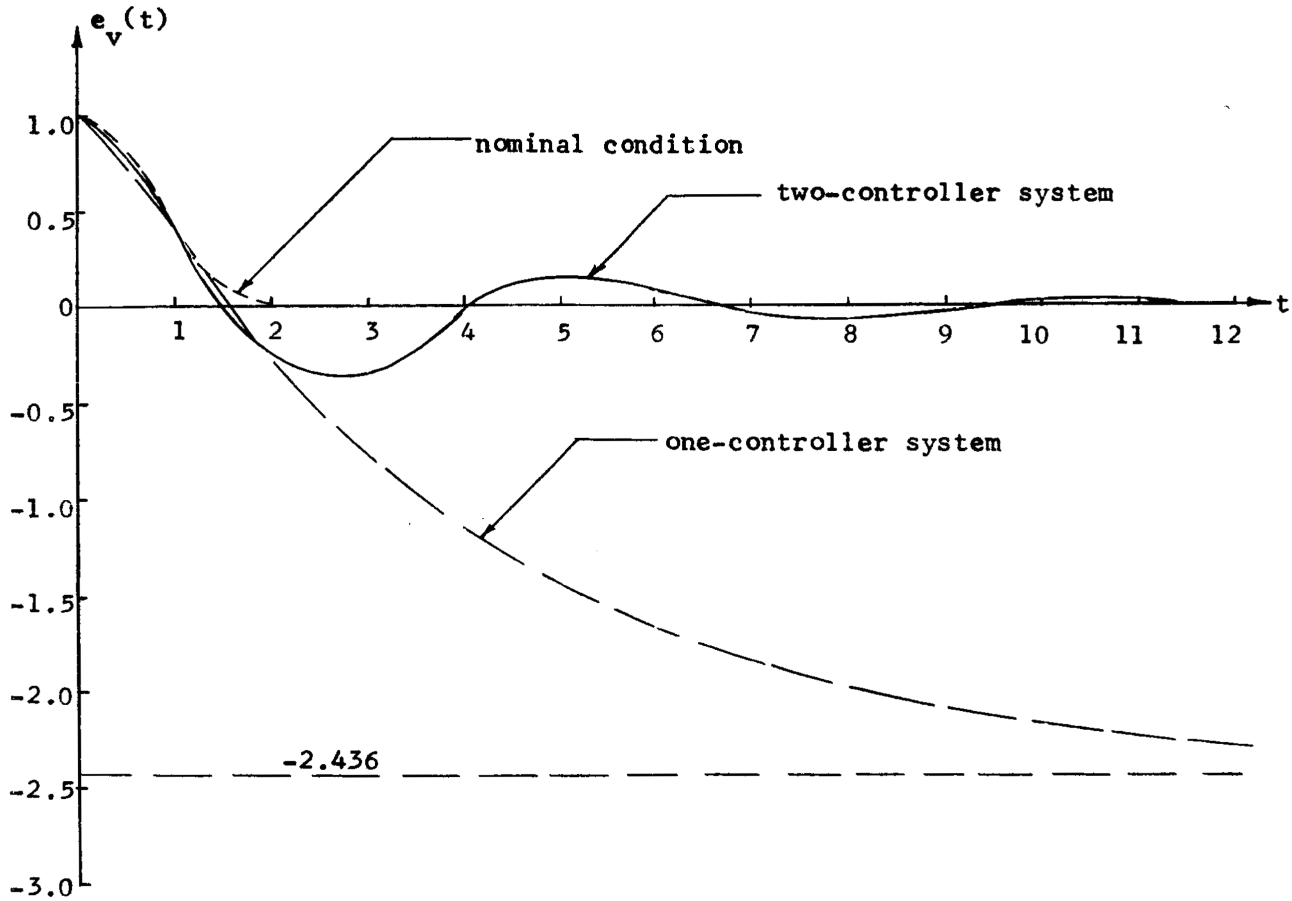


Figure 5. System errors for Example 2.

K_T = 28.2 lb-ft/amp, torque constant ,

K_e = 4.0 volt/rad-sec, emf constant ,

and

R_a = 0.1 ohm, armature resistance .

The motor is preceded by a zero-order hold. Design a deadbeat sampled-data control system with minimal sensitivity in response to step inputs. The sampling period is one second and the varying parameter is the armature circuit resistance, R_a. For example, a 20 percent increase of R_a causes a 18.2 percent decrease in gain, a 2.8 percent decrease in zero and a 10.6 percent decrease in a pole.

Under the nominal condition the following Equations are obtained:

$$G(z) = \frac{0.1092z^{-1}(1 + 0.2639z^{-1})}{(1 - z^{-1})(1 - 0.0104z^{-1})} ,$$

$$H(z) = 0.7912z^{-1}(1 + 0.2639z^{-1}) ,$$

and

$$E(z) = 1 + 0.2088z^{-1} .$$

For two-controller system,

$$D_1(z) = 7.2454$$

and

$$D_2(z) = \frac{1.2770(1 - 0.0103z^{-1})}{1 + 0.2639z^{-1}} .$$

For one-controller system,

$$D(z) = \frac{7.2454(1 - 0.0104z^{-1})}{(1 + 0.2088z^{-1})} \quad .$$

The comparison of the two-controller system to the one-controller system is tabulated in Table III. Figure 6 shows the comparison of system errors.

TABLE III

COMPARISON FOR EXAMPLE 3

20% Increase in R_a

$$G_v(z) = \frac{0.08934z^{-1}(1 + 0.2567z^{-1})}{(1 - z^{-1})(1 - 0.0093z^{-1})}$$

Two-Controller System

$$H_v(z) = \frac{0.6470z^{-1}(1 + 0.2567z^{-1})(1 + 0.2639z^{-1})}{1 + 0.0808z^{-1} - 0.0535z^{-2} + 0.0004z^{-3}}$$

$$E_v(z) = 1 + 0.3530z^{-1} + 0.0685z^{-2} + 0.0130z^{-3} + 0.0025z^{-4} + 0.0005z^{-5} \text{---}$$

One-Controller System

$$H_v(z) = \frac{0.6470z^{-1}(1 + 0.0104z^{-1})(1 + 0.2567z^{-1})}{1 - 0.1535z^{-1} - 0.0420z^{-2} + 0.0002z^{-3}}$$

$$E_v(z) = 1 + 0.3530z^{-1} + 0.0943z^{-2} + 0.0290z^{-3} + 0.0084z^{-4} + 0.0025z^{-5} \text{---}$$

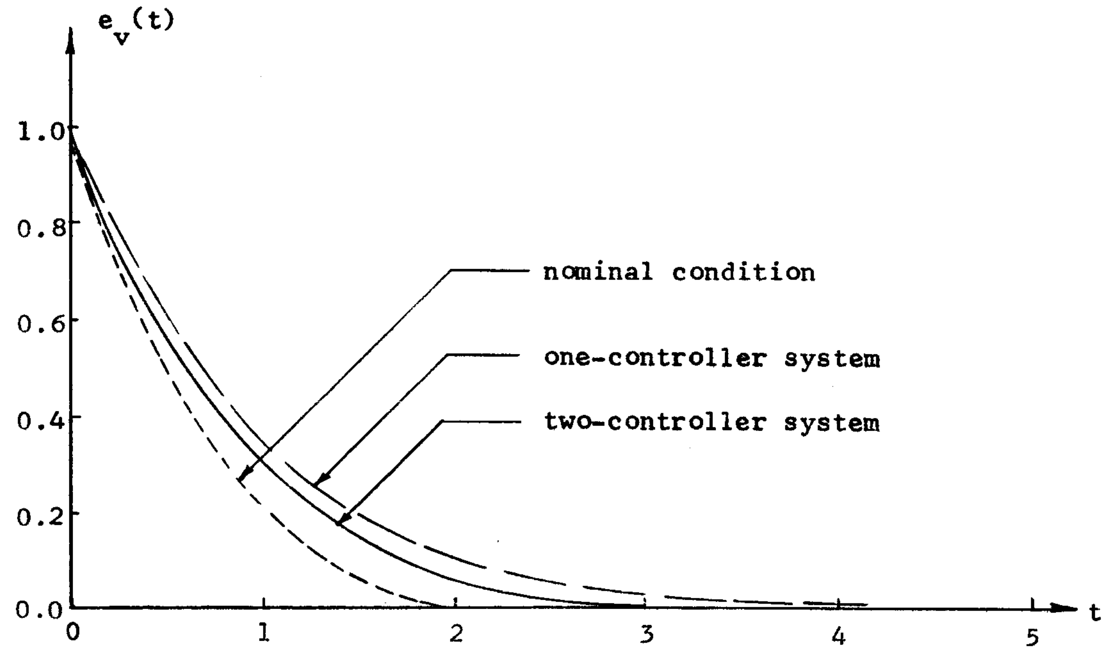


Figure 6. System errors for Example 3.

CHAPTER VI

CONCLUSION

A frequency domain method has been developed for designing the minimal sensitivity deadbeat sampled-data control systems where plant variations are encountered. The systems are designed to have the shortest sensitivity polynomials. The procedure is simple and systematic. Several numerical examples have been given to demonstrate the effectiveness of the method. In general, if the plant variation is not severe, the proposed method gives a two-controller system which is less sensitive than the one-controller system.

It should be mentioned that for most sampled-data control systems, a deadbeat response is not necessary. Furthermore, in general, the plant dynamics are not precisely known, thus a practical deadbeat system is seldom possible. However, the merits of the deadbeat system design are: (1) it results in a general system with fast settling, and (2) the method is simple.

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